MATHCOUNTS® Minis November 2019 Activity Solutions

Warm-Up!

1. An arithmetic sequence with 21 terms (which is an odd number of terms), will have an exact middle term. There will be the first $21 \div 2 = 10.5 \rightarrow 10$ terms, then the 11th term will be the middle term, and then there will be the last 10 terms. The common difference of this arithmetic sequence is 4. To get from the 1st term, 3, to the 11th term, we will need to add the common difference 10 times: $3 + 10(4) =$ **43**.

2. The common difference of this arithmetic sequence is $33 - 31 = 2$, since 31 and 33 are consecutive terms of the sequence. Because this sequence has an even number of terms, 14, there will be the first $14 \div 2 = 7$ terms and the second 7 terms. Therefore, 31 is the 7th term and 33 is the 8th term. To get from the 7th term to the 14th term, we need to add the common difference 7 times: $31 + 7(2) =$ **45**.

3. The 21 integers from -10 through 10 form an arithmetic sequence. The mean of an arithmetic sequence is equal to the median of the sequence. In this case, that value is **0**.

4. Consider a 94% average as 94 out of 100 points on a test. If Ty Wan needs a 94 average for the five tests, then the total number of points scored on the five tests must be at least $5 \times 94 = 470$ points. Ty Wan has already scored $94 + 93 + 96 + 91 = 374$ points. Since $470 - 374 = 96$, Ty Wan needs 96 points or a score of **96**% on the fifth test.

The Problems are solved in the MATHCOUNTS® Mini video.

Follow-up Problems

5. Before the final game, let's say Roberto had bowled g games, and his average score was 205. The sum of the scores from those g games would be $205g$. After the final game, Roberto had bowled a total of $g + 1$ games and increased is average to 209. The sum of the scores from his $g + 1$ games is $209(g + 1)$. Since we are told that Roberto scored 241 in the last game, we know that $209(g + 1) = 205g + 241$. Simplifying and solving for g, we get $209g + 209 = 205g + 241 \rightarrow 4g = 32 \rightarrow g = 8$. So, Roberto had initially bowled 8 games, and the sum of the scores for those 8 games was $205 \times 8 = 1640$. If Roberto's average after bowling a total of $8 + 1 = 9$ games is to be 211, then the sum of his scores would need to be $211 \times 9 = 1899$. Therefore, Roberto would have needed to score $1899 - 1640 =$ **259** points in the final game.

6. If 41 numbers have a sum of 2009, then their mean (average) is $2009 \div 41 = 49$. Since the 41 numbers are in an arithmetic sequence, 49 is also the median of the numbers. That means there are 20 numbers less than 49 and 20 numbers greater than 49. If the constant difference were 1, the sequence would not reach down to single-digit numbers. If the constant difference in the sequence is 2, then the first number would be $20 \times 2 = 40$ less than 49, which is $49 - 9 = 9$. If it were 3, the sequence would go into negative numbers and have several single-digit numbers. Hence, **9** is the only possible one-digit number that could appear in the sequence.

7. If we assume there will only be two consecutive integers with this sum, then we know $210 \div 2 = 105$ is the value between them. However, it's impossible for 105 to be between two consecutive integers. What if we assume we can find three consecutive positive integers with a sum of 210? Then the middle term - and with an odd number of terms, there is an exact middle term - will be $210 \div 3 = 70$, and the terms are 69, 70 and 71, with 71 being the largest. This shows us that with an odd number of terms, that odd number must be a factor of 210 in order to get an exact middle term that is an integer. The odd factors of 210 are 1, 3, 5, 7, 15, 21, 35 and 105. If there were 105 terms, then the middle (53rd) term would be $210 \div 105 = 2$, and we can see that spreading out 52 integers in both directions will result in terms that are negative integers. What if there were 35 terms? The middle (18th) term would be $210 \div 35 = 6$, and again we would get terms that are negative integers when spreading out 17 integers in both directions. Similarly, assuming there are 21 terms will result in a middle (11th) term of 10 and a smallest term of 0. However, with 15 terms, the middle (8th) term is $210 \div 15 = 14$, the smallest term is 7 and the largest term is 21. This takes care of the possibilities with an odd number of terms. However, just because 2 terms didn't work, doesn't mean there couldn't be an even number of terms...

Perhaps there can be four terms... $210 \div 4 = 52.5$. Being that four terms is an even number of terms, this 52.5 must be *exactly between* two consecutive integers, and it is: 52 and 53. The set would be 51, 52, 53 and 54, with 54 being the largest. As we can see, this even number cannot be a factor of 21, or it will result in a mean which is an integer and can't be between two consecutive integers. In fact, the result of dividing 210 by this even number must result in a number that is halfway between two consecutive integers. Is there an even number larger than 15 that works (since we know 15 is the largest we've come up with so far)? After seeing that 16 and 18 don't work, we see 20 does. Notice $210 \div 20 = 10.5$, meaning that the 10th term is 10, the 11th term is 11, the first term is 1 (positive) and the last term is 20. If we try any larger number of even terms, we would run into the problem of getting smaller middle values and extending into the negative integers. The answer then is **20** which results from 20 consecutive integers starting with 1.

8. For the original set of 10 numbers, let's let the average value be x. This tells us the sum of the numbers is $10x$. Removing the largest number results in an average of $x - 1$ and a sum of $9(x - 1) = 9x - 9$ for the 9 remaining numbers. The largest number is then $10x - (9x - 9) = x + 9$. Removing the smallest number results in an average of $x + 2$ and a sum of $9(x + 2) = 9x + 18$. The smallest number is then $10x - (9x + 18) = x - 18$. The positive difference between these two numbers is $(x + 9) - (x - 18) = x + 9 - x + 18 =$ **27**.